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Thermal stresses in an isotropic trimaterial interacted with a pair of point heat source and heat sink

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Abstract

A general analytical solution for an isotropic trimaterial interacted with a point heat source is provided in this paper. Based on the method of analytical continuation in conjunction with the alternating technique, the solutions to heat conduction and thermoelasticity problems for three dissimilar media are first derived. A rapidly convergent series solution for both the temperature and stress functions, which is expressed in terms of an explicit general term of the complex potential of the corresponding homogeneous problem, is obtained in an elegant form. As a numerical illustration, the distributions of thermal stresses along the interface are presented for various material combinations and for different positions of the applied heat source and heat sink.

1. Introduction

Considerable research activities in the area of stress analysis of a layered medium have been carried out in recent years because of the increasing use of composite materials in many engineering applications. Due to the inherent heterogeneous nature of the composites, the analysis of such materials is much more involved than that of homogeneous counterparts. For multilayered composites, the problem becomes more complicated since the solutions are forced to satisfy both the boundary and interface continuity conditions. Consequently, the conventional procedure of stress analysis of multilayered media results in having to solve a system of simultaneous equations for a large number of unknown constants. The complexity of such a procedure can be found in the work of Iyengar and Alwar (1964) as well as Chen (1971) who analyzed the semi-infinite medium composed of isotropic layers. As an alternative efficient approach to the analysis of multilayered media, various solution procedures have been developed. Bufler (1971) used the transfer matrix approach to convert the boundary value problem to an equivalent initial value problem based on the mixed formulation of elasticity proposed by Vlasov and Leontev (1966). This transfer matrix is expressed in

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terms of the infinite series expansion allowing solutions with various orders of approximation to be obtained. Based on the flexibility matrix method, Small and Booker (1984) performed the stress analysis of a layered medium resting on a rigid foundation. This method has been found to have an advantage of significantly reducing the number of simultaneous equations. Lin and Keer (1989) also used the flexibility matrix method together with the boundary integral formulation to deal with a vertical crack in a layered medium. Based on the Fourier transform technique in conjunction with the stiffness matrix approach, Choi and Thangjitham (1991a,b) obtained the solutions of multilayered anisotropic elastic media. Choi and Earmme (2002a,b) employed the alternating technique to obtain the solution of singularity problems in an isotropic and anisotropic trimaterial. All the aforementioned studies are, however, limited to an isothermal condition. When the thermal effect is considered, the problem becomes even more complicated. Padovan (1975, 1976) studied the thermoelastic fields of generally laminated slabs and cylinders subjected to spatially periodic thermal loadings by using the method of complex series expansion. Based on the method of displacement potential, Taucher and Aköz (1975) derived the solutions of thermoelasticity for a simply-supported laminated slab. Tanigawa et al. (1989) performed the transient thermal stress analysis of a laminated composite beam. Choi and Thangjitham (1991a,b) extended the flexibility/stiffness matrix method to the thermoelasticity problem of a multilayered anisotropic medium. To the authors' knowledge, a general analytical solution for the problem of multilayered elastic media interacted with a point heat source has not been found in the open literature.

In this paper, we consider the problem of an isotropic trimaterial interacted with a point heat source. Trimaterial defined here represents an infinite body composed of three dissimilar materials bonded along two parallel interfaces. The proposed method is based on the technique of analytical continuation that is alternatively applied across the two parallel interfaces in order to derive the trimaterial solution in a series form from the corresponding homogeneous solution. A variety of problems such as bimaterial problem, a thin layer bonded to a half-plane, a finite strip of thin film, etc., can be treated as special cases of the present study. The plan of this paper is as follows. The general formulation for plane isotropic thermoelasticity is provided in Section 2. The general forms of the complex potentials of the temperature and stress functions are provided in Sections 3 and 4, respectively. Some special examples are solved in Section 5. Finally, Section 6 concludes the article.

2. Problem formulation

Consider a trimaterial occupying regions $S_a : x_2 \geqslant h$, $S_b : h \geqslant x_2 \geqslant 0$, and $S_c : x_2 \leqslant 0$, respectively, are perfectly bonded along two parallel interfaces $L : x_2 = 0$ and $L^* : x_2 = h$ as shown in Fig. 1. Consider a point heat source of intensity Q_0 located at the point (x_{s1}, x_{s2}) and a point heat sink of intensity $-Q_0$ located at the point (x_{k1}, x_{k2}) that may cause a thermal stress distribution as a result of the different thermoelastic properties of the three phases. For a two-dimensional heat conduction problem, the resultant heat flow Q and the temperature T can be expressed in terms of a single complex potential $g'(z)$ as

$$Q = \int (q_{x_1}\,\mathrm{d}y - q_{x_2}\,\mathrm{d}x) = -k\,\mathrm{Im}[g'(z)] \tag{1}$$

$$T = \mathrm{Re}[g'(z)] \tag{2}$$

where Re and Im denote the real part and imaginary part of the bracketed expression, respectively and primes denote differentiation with respect to $z(z = x_1 + \mathrm{i}x_2)$. The quantities q_{x_1}, q_{x_2} in Eq. (1) are the components of heat flux in the x_1 and x_2 direction, respectively, k stands for the heat conductivity. Once the heat conduction problem is solved, the temperature function $g'(z)$ is determined. For a two-dimensional

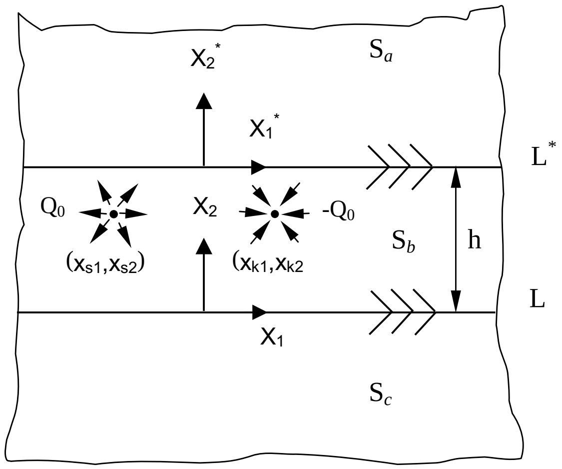

Fig. 1. A pair of point heat source and sink in a trimaterial.

theory of thermoelasticity, the components of the displacements and stresses can be expressed in terms of two stress functions $\Phi(z)$, $\Omega(z)$ and a temperature function $g'(z)$ as

$$2G\frac{\partial}{\partial x_1}(u_1 + \mathrm{i}u_2) = \kappa\Phi(z) - \overline{\Omega(z)} - (z - \bar{z})\overline{\Phi'(z)} + 2G\beta g'(z) \tag{3}$$

$$\sigma_{22} - \mathrm{i}\sigma_{12} = \Phi(z) + \overline{\Omega(z)} + (z - \bar{z})\overline{\Phi'(z)} \tag{4}$$

where G is the shear modulus, $\kappa = 3 - 4v$, $\beta = (1 + v)\alpha$ for plane strain and $(3 - v)/(1 + v)$, $\beta = \alpha$ for plane stress with v being the Poisson's ratio and α the thermal expansion coefficient. Here a superimposed bar represents the complex conjugate.

For the problem associated with an isotropic elastic bimaterial, the stresses are found to depend on only two non-dimensional Dundurs parameters (Dundurs, 1969)

$$\alpha_{ab} = \frac{G_a(\kappa_b + 1) - G_b(\kappa_a + 1)}{G_a(\kappa_b + 1) + G_b(\kappa_a + 1)}, \qquad \beta_{ab} = \frac{G_a(\kappa_b - 1) - G_b(\kappa_a - 1)}{G_a(\kappa_b + 1) + G_b(\kappa_a + 1)} \tag{5}$$

where a and b refer to the two materials composing the bimaterial. Another pairs associated with the above two parameters are defined as

$$\Lambda_{ab} = \frac{\alpha_{ab} + \beta_{ab}}{1 - \beta_{ab}}, \qquad \Pi_{ab} = \frac{\alpha_{ab} - \beta_{ab}}{1 + \beta_{ab}} \tag{6}$$

which will be used in our subsequent derivations for trimaterial problems.

3. Temperature function

3.1. A singularity embedded in S_b

To obtain the thermal potential $\theta(z) = g'(z)$ for the problem with a singularity in a trimaterial with two parallel interfaces as shown in Fig. 1, the alternating technique is applied.

Step 1: Analytical continuation across the interface L

First, we regard regions S_a and S_b composed of the same material b and region S_c of material c. If $\theta_0(z)$ signifies a potential for a singularity in an infinite homogeneous plane of material b, $\theta_{c0}(z)$ analytical in S_c and $\theta_1(z)$ analytical in $S_a \cup S_b$ are introduced to satisfy the continuity conditions across L as

$$\theta(z) = \begin{cases} \theta_0(z) + \theta_1(z) & z \in S_a \cup S_b \\ \theta_{c0}(z) & z \in S_c \end{cases} \tag{7}$$

The continuity of resultant heat flow and temperature across the interface L requires

$$\begin{cases} \theta_{c0}(x_1) + \overline{\theta_{c0}}(x_1) = \theta_1(x_1) + \overline{\theta_1}(x_1) + \theta_0(x_1) + \overline{\theta_0}(x_1) \\ k_c[\theta_{c0}(x_1) - \overline{\theta_{c0}}(x_1)] = k_b\{[\theta_0(x_1) + \theta_1(x_1)] - [\overline{\theta_0}(x_1) + \overline{\theta_1}(x_1)]\} \end{cases} \tag{8}$$

By the standard analytic continuation arguments it follows that

$$\begin{cases} \overline{\theta_{c0}}(z) = \theta_1(z) + \overline{\theta_0}(z) & z \in S_a \cup S_b \\ \theta_{c0}(z) = \overline{\theta_1}(z) + \theta_0(z) & z \in S_c \end{cases} \tag{9}$$

$$\begin{cases} k_c\overline{\theta_{c0}}(z) = k_b\overline{\theta_0}(z) - k_b\theta_1(z) & z \in S_a \cup S_b \\ k_c\theta_{c0}(z) = k_b\theta_0(z) - k_b\overline{\theta_1}(z) & z \in S_c \end{cases} \tag{10}$$

With Eqs. (9) and (10) one obtains

$$\begin{cases} \theta_1(z) = V_{cb}\overline{\theta_0}(z) & z \in S_a \cup S_b \\ \theta_{c0}(z) = U_{cb}\theta_0(z) & z \in S_c \end{cases} \tag{11}$$

where

$$\begin{aligned} U_{cb} &= 2k_b(k_c + k_b)^{-1} \\ V_{cb} &= (k_b - k_c)(k_c + k_b)^{-1} \end{aligned} \tag{12}$$

Since this result is based on the assumption that region S_a is made up of material b, it cannot satisfy the continuity conditions at the interface L^* which lies between material a and b.

Step 2: Analytical continuation across the interface L^*

Nest, we assume regions S_b and S_c be made up of the same material b and region S_a of material a. Additional terms $\theta_{b1}(z)$ analytical in $S_b \cup S_c$ and $\theta_{a1}(z)$ analytical in S_a are introduced to satisfy the continuity conditions across the interface L^* that

$$\begin{cases} \theta^*_{a1}(x^*_1) + \overline{\theta^*_{a1}}(x^*_1) = \theta^*_1(x^*_1) + \overline{\theta^*_1}(x^*_1) + \theta_0(x^*_1) + \overline{\theta^*_0}(x^*_1) + \theta^*_{b1}(x^*_1) + \overline{\theta^*_{b1}(x^*_1)} \\ k_a[\theta^*_{a1}(x^*_1) - \overline{\theta^*_{a1}}(x^*_1)] = k_b\{[\theta^*_0(x^*_1) + \theta^*_1(x^*_1) + \theta^*_{b1}(x^*_1)] - [\overline{\theta^*_0}(x^*_1) + \overline{\theta^*_1}(x^*_1) + \overline{\theta^*_{b1}}(x^*_1)]\} \end{cases} \tag{13}$$

Here, ‘*’ denotes the field in $x^*_1\ x^*_2$ system. By analytical continuation method one can obtain

$$\begin{cases} \theta^*_{a1}(z^*) = U_{ab}[\theta^*_1(z^*) + \theta^*_0(z^*)] & z^* \in S_a \\ \theta^*_{b1}(z) = V_{ab}\left[\overline{\theta^*_1}(z^*) + \overline{\theta^*_0}(z^*)\right] & z^* \in S_b \cup S_c \end{cases} \tag{14}$$

where U_{ab} and V_{ab} are defined as in Eq. (12).

With a coordinate translation $z^* = z - \mathrm{i}h$ (see Fig. 1), it is easy to show that the thermal potential $\theta(z)$ in the x_1x_2 coordinate system is related to the function $\theta^*(z^*)$ in the $x^*_1x^*_2$ coordinate system by

$$\theta^*(z^*) = \theta(z), \qquad \overline{\theta^*}(z^*) = \overline{\theta}(z - 2\mathrm{i}h) \tag{15}$$

Substitution of Eq. (15) to Eq. (14) yields

$$\begin{cases} \theta_{a1}(z) = U_{ab}[\theta_1(z) + \theta_0(z)] & z \in S_a \\ \theta_{b1}(z) = V_{ab}\left[\overline{\theta_1}(z-2\mathrm{i}h) + \overline{\theta_0}(z-2\mathrm{i}h)\right] & z \in S_b \cup S_c \end{cases} \tag{16}$$

Since this result is based on the assumption that region S_c is made up of material b, it cannot satisfy the continuity conditions at the interface L.

Step 3: Analytical continuation across the interface L

We again assume regions S_a and S_b be made up of the same material b and region S_c of material c. Additional terms $\theta_2(z)$ analytical in $S_a \cup S_b$ and $\theta_{c1}(z)$ analytical in S_c are introduced to satisfy the continuity conditions across the interface L. By a similar way to the previous approach, one can find

$$\begin{cases} \theta_2(z) = V_{cb}\overline{\theta_{b1}}(z) & z \in S_a \cup S_b \\ \theta_{c1}(z) = U_{cb}\theta_{b1}(z) & z \in S_c \end{cases} \tag{17}$$

Obviously, this result cannot satisfy the continuity conditions at the interface L^*.

Step 4: Repetitions of steps 2 and 3

The method of analytical continuation is repeatedly performed across the two interfaces to achieve the additional terms $\theta_{an}(z)$, $\theta_{bn}(z)$, $\theta_{cn}(z)$, $\theta_{n+1}(z)$ for $n = 2, 3, \ldots$ Consequently, one can find the complete solution of $\theta(z)$ as

$$\theta(z) = \begin{cases} \sum_{n=1}^{\infty}\theta_{an}(z) & z \in S_a \\ \theta_0(z) + \sum_{n=1}^{\infty}\theta_n(z) + \sum_{n=1}^{\infty}\theta_{bn}(z) & z \in S_b \\ \theta_{c0}(z) + \sum_{n=1}^{\infty}\theta_{cn}(z) & z \in S_c \end{cases} \tag{18}$$

Since $\theta_{an}(z)$, $\theta_{bn}(z)$ and $\theta_{cn}(z)$ can be expressed in term of $\theta_0(z)$, Eq. (18) becomes

$$\theta(z) = \begin{cases} U_{ab}\theta_0(z) + U_{ab}\sum_{n=1}^{\infty}\theta_n(z) & z \in S_a \\ \theta_0(z) + \sum_{n=1}^{\infty}\theta_n(z) + V_{ab}\overline{\theta_0}(z-2\mathrm{i}h) + V_{ab}\sum_{n=1}^{\infty}\overline{\theta_n}(z-2\mathrm{i}h) & z \in S_b \\ U_{cb}\theta_0(z) + U_{cb}V_{ab}\overline{\theta_0}(z-2\mathrm{i}h) + U_{cb}V_{ab}\sum_{n=1}^{\infty}\overline{\theta_n}(z-2\mathrm{i}h) & z \in S_c \end{cases} \tag{19}$$

where the recurrence formulae for $\theta_n(z)$ is

$$\theta_{n+1}(z) = \begin{cases} V_{cb}\overline{\theta_0}(z) & n = 0 \\ V_{cb}V_{ab}[\theta_n(z+2\mathrm{i}h) + \theta_0(z+2\mathrm{i}h)] & n = 1 \\ V_{cb}V_{ab}\theta_n(z+2\mathrm{i}h) & n = 2, 3, 4, \ldots \end{cases} \tag{20}$$

For a point heat source of intensity Q_0 located in the point $z = z_s$ and a point heat sink of the same intensity located in the point $z = z_k$ in the infinite homogeneous plate, the solution is

$$\theta_0(z) = -\frac{Q_0}{2\pi k}\log\left(\frac{z - z_s}{z - z_k}\right) \tag{21}$$

3.2. A singularity embedded in S_c

Using the same procedure as Section 3.1, the solution of the other case in which the singularity is located in region S_c is

$$\theta(z)=\begin{cases} U_{ab}\sum_{n=1}^{\infty}\theta_n(z) & z\in S_a \\ \sum_{n=1}^{\infty}\theta_n(z)+V_{ab}\sum_{n=1}^{\infty}\overline{\theta_n}(z-2\mathrm{i}h) & z\in S_b \\ \theta_0(z)+V_{bc}\overline{\theta_0}(z)+U_{cb}V_{ab}\sum_{n=1}^{\infty}\overline{\theta_n}(z-2\mathrm{i}h) & z\in S_c \end{cases} \tag{22}$$

where the recurrence formulae for $\theta_n(z)$ is

$$\theta_{n+1}(z)=\begin{cases} U_{bc}\theta_0(z) & n=0 \\ V_{cb}V_{ab}\theta_n(z+2\mathrm{i}h) & n=1,2,3,\ldots \end{cases} \tag{23}$$

4. Stress function

4.1. A coordinate translation

Suppose that the region $S_a : x_2 \geqslant h$ and $S_b \leqslant h$ occupied by material a and b, respectively, are perfectly bonded along the interface $x_2 = h$. With a coordinate translation $z^* = z - \mathrm{i}h$ (see Fig. 1 with material c = material b), the potentials $\Phi(z)$ and $\Omega(z)$ in the x_1x_2 coordinate system are related to the potentials $\Phi^*(z^*)$ and $\Omega^*(z^*)$ in the $x_1^*\,x_2^*$ coordinate system by

$$\Phi(z)=\Phi^*(z^*),\qquad \Omega(z)=\Omega^*(z^*)+2\mathrm{i}h\Phi'^*(z^*) \tag{24}$$

With $\overline{\Phi^*}(z^*)=\overline{\Phi}(z-2\mathrm{i}h)$, it is easy to show

$$\overline{\Omega^*}(z^*)=\overline{\Omega}(z-2\mathrm{i}h)+2\mathrm{i}h\overline{\Phi'}(z-2\mathrm{i}h) \tag{25}$$

4.2. A singularity embedded in S_b

We first regard regions S_a and S_b composed of the same material b and region S_c of material c. If $\Phi_{bh}(z)$ and $\Omega_{bh}(z)$ signify the stress functions for a singularity in an infinite homogeneous plane of material b, $\Phi_{ba}(z)$ and $\Omega_{ba}(z)$ analytical in $S_a \cup S_b$, $\Phi_{cc}(z)$ and $\Omega_{cc}(z)$ analytical in S_c are introduced to satisfy the continuity conditions across L as

$$\begin{cases} \Phi(z)=\Phi_{ba}(z)+\Phi_{bh}(z) & z\in S_a\cup S_b \\ \Phi(z)=\Phi_{cc}(z) & z\in S_c \end{cases} \tag{26a}$$

$$\begin{cases} \Omega(z)=\Omega_{ba}(z)+\Omega_{bh}(z) & z\in S_a\cup S_b \\ \Omega(z)=\Omega_{cc}(z) & z\in S_c \end{cases} \tag{26b}$$

The continuity of traction and displacement across L yields

$$\Phi_{cc}(x_1)+\overline{\Omega_{cc}}(x_1)=\Phi_{ba}(x_1)+\Phi_{bh}(x_1)+\overline{\Omega_{ba}}(x_1)+\overline{\Omega_{bh}}(x_1) \tag{27a}$$

$$\frac{1}{2G_c}\left[\kappa_c\Phi_{cc}(x_1)-\overline{\Omega_{cc}}(x_1)\right]+\beta_c\theta_{cc}(x_1)=\frac{1}{2G_b}\left[\kappa_b\Phi_{ba}(x_1)+\kappa_b\Phi_{bh}(x_1)-\overline{\Omega_{ba}}(x_1)-\overline{\Omega_{bh}}(x_1)\right]+\beta_b[\theta_0(x_1)+\theta_{ba}(x_1)+\theta_{bc}(x_1)] \tag{27b}$$

By the standard analytical continuation arguments it follows that

$$
\begin{aligned}
&\Phi_{cc}(z) = \overline{\Omega_{ba}}(z) + \Phi_{bh}(z) \quad z \in S_c \\
&\overline{\Omega_{bh}}(z) + \Phi_{ba}(z) = \overline{\Omega_{cc}}(z) \quad z \in S_a \cup S_b \\
&\frac{-\overline{\Omega_{cc}}(z)}{2G_c} = \frac{\kappa_b \Phi_{ba}(z) - \overline{\Omega_{bh}}(z)}{2G_b} + \beta_b \theta_{ba}(z) \quad z \in S_a \cup S_b
\end{aligned} \tag{28a}
$$

$$
\frac{\kappa_c \Phi_{cc}(z)}{2G_c} + \beta_c \theta_{cc}(z) = \frac{\kappa_b \Phi_{bh}(z) - \overline{\Omega_{ba}}(z)}{2G_b} + \beta_b[\theta_{bc}(z) + \theta_0(z)] \quad z \in S_c \tag{28b}
$$

From Eqs. (28a) and (28b), one can find

$$
\begin{aligned}
&\Phi_{ba}(z) = \frac{-\beta_b \theta_{ba}(z) + \left(\frac{1}{2G_b} - \frac{1}{2G_c}\right)\overline{\Omega_{bh}}(z)}{\frac{\kappa_b}{2G_b} + \frac{1}{2G_c}} \quad z \in S_a \cup S_b \\
&\Phi_{cc}(z) = \frac{\beta_b[\theta_{bc}(z) + \theta_0(z)] - \beta_c \theta_{cc}(z) + \frac{(1+\kappa_b)}{2G_b}\Phi_{bh}(z)}{\frac{\kappa_c}{2G_c} + \frac{1}{2G_b}} \quad z \in S_c \\
&\Omega_{ba}(z) = \frac{\beta_b\left[\overline{\theta_{bc}}(z) + \overline{\theta_0}(z)\right] - \beta_c \overline{\theta_{cc}}(z) + \left(\frac{\kappa_b}{2G_b} - \frac{\kappa_c}{2G_c}\right)\overline{\Phi_{bh}}(z)}{\frac{\kappa_c}{2G_c} + \frac{1}{2G_b}} \quad z \in S_a \cup S_b
\end{aligned} \tag{29a}
$$

$$
\Omega_{cc}(z) = \frac{-\beta_b \overline{\theta_{ba}}(z) + \frac{(\kappa_b+1)}{2G_b}\Omega_{bh}(z)}{\frac{\kappa_b}{2G_b} + \frac{1}{2G_c}} \quad z \in S_c \tag{29b}
$$

where

$$
\Phi_{bh}(z) = \frac{G_b Q_0 \beta_b}{4\pi k_b(1-v_b)} \log\left(\frac{z - z_s}{z - z_k}\right) \tag{30}
$$

$$
\Omega_{bh}(z) = \frac{G_b Q_0 \beta_b}{4\pi k_b(1-v_b)} \left[\log\left(\frac{z - z_s}{z - z_k}\right) + \frac{z - \overline{z_s}}{z - z_s} - \frac{z - \overline{z_k}}{z - z_k}\right] \tag{31}
$$

$$
\theta_{ba}(z) = \sum_{n=1}^{\infty} \theta_n(z) \tag{32}
$$

$$
\theta_{bc}(z) = V_{ab}\overline{\theta_0}(z - 2\mathrm{i}h) + V_{ab}\sum_{n=1}^{\infty}\overline{\theta_n}(z - 2\mathrm{i}h) \tag{33}
$$

$$
\theta_{cc}(z) = U_{cb}\theta_0(z) + U_{cb}V_{ab}\overline{\theta_0}(z - 2\mathrm{i}h) + U_{cb}V_{ab}\sum_{n=1}^{\infty}\overline{\theta_n}(z - 2\mathrm{i}h) \tag{34}
$$

Since this result is based on the assumption that region S_a is made up of material b, it cannot satisfy the continuity condition across L^*.

Next, we assume regions S_b and S_c be made up of the same material b and region S_a of material a. Additional terms $\Phi_0(z)$ and $\Omega_0(z)$ analytical in $S_b \cup S_c$, $\Phi_{aa}(z)$ and $\Omega_{aa}(z)$ analytical in S_a are introduced to satisfy the continuity conditions across L^* as

$$
\Phi_{aa}^*(x_1^*) + \overline{\Omega_{aa}^*}(x_1^*) = \Phi_{ba}^*(x_1^*) + \overline{\Omega_{ba}^*}(x_1^*) + \Phi_{bh}^*(x_1^*) + \overline{\Omega_{bh}^*}(x_1^*) + \Phi_0^*(x_1^*) + \overline{\Omega_0^*}(x_1^*) \tag{35a}
$$

$$\frac{1}{2G_a}\left[\kappa_a\Phi^*_{aa}(x_1^*)-\overline{\Omega^*_{aa}}(x_1^*)\right]+\beta_a\theta^*_{aa}(x_1^*)=\frac{1}{2G_b}[\kappa_b\Phi^*_{ba}(x_1^*)+\kappa_b\Phi^*_{bh}(x_1^*)+\kappa_b\Phi^*_0(x_1^*)-\overline{\Omega^*_{ba}}(x_1^*)-\overline{\Omega^*_{bh}}(x_1^*)-\overline{\Omega^*_0}(x_1^*)]+\beta_b[\theta^*_0(x_1^*)+\theta^*_{ba}(x_1^*)+\theta^*_{bc}(x_1^*)] \quad (35b)$$

where

$$\theta_{aa}(z)=U_{ab}\theta_0(z)+U_{ab}\sum_{n=1}^{\infty}\theta_n(z) \quad (36)$$

By the standard analytical continuation arguments it follows that

$$\begin{aligned}&\Phi^*_{aa}(z^*)=\Phi^*_{ba}(z^*)+\Phi^*_{bh}(z^*)+\overline{\Omega^*_0}(z^*) \quad z^*\in S_a\\&\overline{\Omega^*_{aa}}(z^*)=\Phi^*_0(z^*)+\overline{\Omega^*_{ba}}(z^*)+\overline{\Omega^*_{bh}}(z^*) \quad z^*\in S_b\cup S_c\end{aligned} \quad (37a)$$

$$\begin{aligned}&\frac{\kappa_a\Phi^*_{aa}(z^*)}{2G_a}+\beta_a\theta^*_{aa}(z^*)=\frac{\kappa_b\Phi^*_{ba}(z^*)+\kappa_b\Phi^*_{bh}(z^*)-\overline{\Omega^*_0}(z^*)}{2G_b}+\beta_b[\theta^*_0(z^*)+\theta^*_{ba}(z^*)] \quad z^*\in S_a\\&-\frac{\overline{\Omega^*_{aa}}(z^*)}{2G_a}=\frac{\kappa_b\Phi^*_0(z^*)-\overline{\Omega^*_{ba}}(z^*)-\overline{\Omega^*_{bh}}(z^*)}{2G_b}+\beta_b\theta^*_{bc}(z^*) \quad z^*\in S_b\cup S_c\end{aligned} \quad (37b)$$

From Eqs. (37a) and (37b), one can find

$$\begin{aligned}&\Phi^*_0(z^*)=\frac{\left(\frac{1}{G_b}-\frac{1}{G_a}\right)[\overline{\Omega^*_{ba}}(z^*)+\overline{\Omega_{bh}}^*(z^*)]-2\beta_b\theta^*_{bc}(z^*)}{\frac{1}{G_a}+\frac{\kappa_b}{G_b}} \quad z^*\in S_b\cup S_c\\&\Phi^*_{aa}(z^*)=\frac{\frac{\kappa_b+1}{G_b}[\Phi^*_{ba}(z^*)+\Phi^*_{bh}(z^*)]+2\beta_b[\theta^*_0(z^*)+\theta^*_{ba}(z^*)]-2\beta_a\theta^*_{aa}(z^*)}{\frac{\kappa_a}{G_a}+\frac{1}{G_b}} \quad z^*\in S_a\end{aligned} \quad (38a)$$

$$\begin{aligned}&\Omega^*_0(z^*)=\frac{\left(\frac{\kappa_b}{G_b}-\frac{\kappa_a}{G_a}\right)[\overline{\Phi^*_{ba}}(z^*)+\overline{\Phi^*_{bh}}(z^*)]+2\beta_b[\overline{\theta^*_0}(z^*)+\overline{\theta^*_{ba}}(z^*)]-2\beta_a\overline{\theta^*_{aa}}(z^*)}{\frac{\kappa_a}{G_a}+\frac{1}{G_b}} \quad z^*\in S_b\cup S_c\\&\Omega^*_{aa}(z^*)=\frac{\left(\frac{1+\kappa_b}{G_b}\right)[\Omega^*_{ba}(z^*)+\Omega^*_{bh}(z^*)]-2\beta_b\overline{\theta^*_{bc}}(z^*)}{\frac{1}{G_a}+\frac{\kappa_b}{G_b}} \quad z^*\in S_a\end{aligned} \quad (38b)$$

With the aid of Eqs. (15), (24) and (25), Eqs. (38a) and (38b) become

$$\begin{aligned}&\Phi_0(z)=\frac{\left(\frac{1}{G_b}-\frac{1}{G_a}\right)[\overline{\Omega_{ba}}(z-2ih)+2ih\overline{\Phi'_{ba}}(z-2ih)]-2\beta_b\theta_{bc}(z)}{\frac{1}{G_a}+\frac{\kappa_b}{G_b}}\\&\qquad+\frac{\left(\frac{1}{G_b}-\frac{1}{G_a}\right)[\overline{\Omega_{bh}}(z-2ih)+2ih\overline{\Phi'_{bh}}(z-2ih)]}{\frac{1}{G_a}+\frac{\kappa_b}{G_b}} \quad z\in S_b\cup S_c\\&\Phi_{aa}(z)=\frac{\frac{(\kappa_b+1)}{G_b}[\Phi_{ba}(z)+\Phi_{bh}(z)]+2\beta_b[\theta_0(z)+\theta_{ba}(z)]-2\beta_a\theta_{aa}(z)}{\frac{\kappa_a}{G_a}+\frac{1}{G_b}} \quad z\in S_a\end{aligned} \quad (39a)$$

$$\Omega_0(z) = 2\mathrm{i}h\Phi_0'(z) + \frac{\left(\frac{\kappa_b}{G_b} - \frac{\kappa_a}{G_a}\right)[\overline{\Phi_{ba}}(z-2\mathrm{i}h) + \overline{\Phi_{bh}}(z-2\mathrm{i}h)] + 2\beta_b[\overline{\theta_0}(z-2\mathrm{i}h) + \overline{\theta_{ba}}(z-2\mathrm{i}h)] - 2\beta_a\overline{\theta_{aa}}(z-2\mathrm{i}h)}{\frac{\kappa_a}{G_a} + \frac{1}{G_b}}$$

$$z \in S_b \cup S_c$$

$$\Omega_{aa}(z) = 2\mathrm{i}h\Phi_{aa}'(z) + \frac{\frac{\kappa_b+1}{G_b}[\Omega_{ba}(z) - 2\mathrm{i}h\Phi_{ba}'(z) + \Omega_{bh}(z) - 2\mathrm{i}h\Phi_{bh}'(z)] - 2\beta_b\overline{\theta_{bc}}(z-2\mathrm{i}h)}{\frac{1}{G_a} + \frac{\kappa_b}{G_b}} \quad z \in S_a \tag{39b}$$

Since this result is based on the assumption that region S_c is made up of material b, it cannot satisfy the continuity conditions across L. The procedures are similar to the previous approach and therefore the details are suppressed here. The final results are as follows

$$\Phi(z) = \begin{cases} \Phi_{aa}(z) + (1+\Lambda_{ab})\sum_{n=1}^{\infty}\Phi_n(z) & z \in S_a \\ \Phi_{ba}(z) + \Phi_{bh}(z) + \Phi_0(z) + \sum_{n=1}^{\infty}[\Phi_n(z) + \Lambda_{cb}^{-1}\overline{\Omega}_{n+1}(z)] & z \in S_b \\ \Phi_{cc}(z) + (1+\Lambda_{cb})\Phi_0(z) + (1+\Lambda_{cb}^{-1})\sum_{n=1}^{\infty}\overline{\Omega}_{n+1}(z) & z \in S_c \end{cases} \tag{40a}$$

$$\Omega(z) = \begin{cases} \Omega_{aa}(z) + (1+\Pi_{ab})\sum_{n=1}^{\infty}\Omega_n(z) + 2\mathrm{i}h(\Lambda_{ab} - \Pi_{ab})\sum_{n=1}^{\infty}\Phi_n'(z) & z \in S_a \\ \Omega_{ba}(z) + \Omega_{bh}(z) + \Omega_0(z) + \sum_{n=1}^{\infty}[\Omega_n(z) + \Pi_{cb}^{-1}\overline{\Phi}_{n+1}(z)] & z \in S_b \\ \Omega_{cc}(z) + (1+\Pi_{cb})\Omega_0(z) + (1+\Pi_{cb}^{-1})\sum_{n=1}^{\infty}\overline{\Phi}_{n+1}(z) & z \in S_c \end{cases} \tag{40b}$$

where the recurrence formulae for $\Phi_n(z)$ and $\Omega_n(z)$ are

$$\Phi_{n+1}(z) = \begin{cases} \Pi_{cb}\overline{\Omega_0}(z) & \text{for } n = 0 \\ \Pi_{cb}[\Lambda_{ab}\Phi_n(z+2\mathrm{i}h) - 2\mathrm{i}h\Pi_{ab}\Omega_n'(z+2\mathrm{i}h) - 4h^2\Pi_{ab}\Phi_n''(z+2\mathrm{i}h)] & \text{for } n = 1,2,3,\ldots \end{cases} \tag{41a}$$

$$\Omega_{n+1}(z) = \begin{cases} \Lambda_{cb}\overline{\Phi_0}(z) & \text{for } n = 0 \\ \Pi_{ab}\Lambda_{cb}[\Omega_n(z+2\mathrm{i}h) - 2\mathrm{i}h\Phi_n'(z+2\mathrm{i}h)] & \text{for } n = 1,2,3,\ldots \end{cases} \tag{41b}$$

For the special case that material a is made up of material b, Eq. (40) reduces to

$$\Phi(z) = \begin{cases} \dfrac{-\beta_b V_{cb}\overline{\theta_0}(z) + \left(\frac{1}{2G_b} - \frac{1}{2G_c}\right)\overline{\Omega_{bh}}(z)}{\frac{\kappa_b}{2G_b} + \frac{1}{2G_c}} + \Phi_{bh}(z) & z \in S_b \\ \dfrac{(\beta_b - \beta_c U_{cb})\theta_0(z) + \frac{(1+\kappa_b)}{2G_b}\Phi_{bh}(z)}{\frac{\kappa_c}{2G_c} + \frac{1}{2G_b}} & z \in S_c \end{cases} \tag{42a}$$

$$\Omega(z) = \begin{cases} \dfrac{(\beta_b - \beta_c U_{cb})\overline{\theta_0}(z) + \left(\frac{\kappa_b}{2G_b} - \frac{\kappa_c}{2G_c}\right)\overline{\Phi_{bh}}(z)}{\frac{\kappa_c}{2G_c} + \frac{1}{2G_b}} + \Omega_{bh}(z) & z \in S_b \\ \dfrac{-\beta_b V_{cb}\theta_0(z) + \frac{(\kappa_b+1)}{2G_b}\Omega_{bh}(z)}{\frac{\kappa_b}{2G_b} + \frac{1}{2G_c}} & z \in S_c \end{cases} \tag{42b}$$

which is in agreement with the exact solution of the bimaterial one (Chao and Shen, 1993).

4.3. A singularity embedded in S_c

By the same arguments as Section 4.2, the solution of the other case in which the singularity is located in region S_c is

$$\Phi(z) = \begin{cases} \Phi_{aa}(z) + (1+\Lambda_{ab})\sum_{n=1}^{\infty}\Phi_n(z) & z \in S_a \\ \Phi_{ba}(z) + \Phi_0(z) + \sum_{n=1}^{\infty}[\Phi_n(z) + \Lambda_{cb}^{-1}\overline{\Omega}_{n+1}(z)] & z \in S_b \\ \Phi_{cc}(z) + \Phi_{ch}(z) + (1+\Lambda_{cb})\Phi_0(z) + (1+\Lambda_{cb}^{-1})\sum_{n=1}^{\infty}\overline{\Omega}_{n+1}(z) & z \in S_c \end{cases} \tag{43a}$$

$$\Omega(z) = \begin{cases} \Omega_{aa}(z) + (1+\Pi_{ab})\sum_{n=1}^{\infty}\Omega_n(z) + 2ih(\Lambda_{ab} - \Pi_{ab})\sum_{n=1}^{\infty}\Phi'_n(z) & z \in S_a \\ \Omega_{ba}(z) + \Omega_0(z) + \sum_{n=1}^{\infty}[\Omega_n(z) + \Pi_{cb}^{-1}\overline{\Phi}_{n+1}(z)] & z \in S_b \\ \Omega_{cc}(z) + \Omega_{ch}(z) + (1+\Pi_{cb})\Omega_0(z) + (1+\Pi_{cb}^{-1})\sum_{n=1}^{\infty}\overline{\Phi}_{n+1}(z) & z \in S_c \end{cases} \tag{43b}$$

where the recurrence formulae for $\Phi_n(z)$ and $\Omega_n(z)$ are

$$\Phi_{n+1}(z) = \begin{cases} \Pi_{cb}\overline{\Omega_0}(z) & \text{for } n = 0 \\ \Pi_{cb}[\Lambda_{ab}\Phi_n(z+2ih) - 2ih\Pi_{ab}\Omega'_n(z+2ih) - 4h^2\Pi_{ab}\Phi''_n(z+2ih)] & \text{for } n = 1,2,3,\ldots \end{cases} \tag{44a}$$

$$\Omega_{n+1}(z) = \begin{cases} \Lambda_{cb}\overline{\Phi_0}(z) & \text{for } n = 0 \\ \Pi_{ab}\Lambda_{cb}[\Omega_n(z+2ih) - 2ih\Phi'_n(z+2ih)] & \text{for } n = 1,2,3,\ldots \end{cases} \tag{44b}$$

and

$$\Phi_{ba}(z) = \frac{\left(\frac{1+\kappa_c}{2G_c}\right)\Phi_{ch}(z) - \beta_b\theta_{ba}(z) + \beta_c\theta_0(z)}{\frac{\kappa_b}{2G_b} + \frac{1}{2G_c}}$$
$$\Phi_{cc}(z) = \frac{\left(\frac{1}{2G_c} - \frac{1}{2G_b}\right)\overline{\Omega_{ch}}(z) + \beta_b\theta_{bc}(z) - \beta_c\theta_{cc}(z)}{\frac{\kappa_c}{2G_c} + \frac{1}{2G_b}} \tag{45a}$$

$$\Omega_{ba}(z) = \frac{\left(\frac{1+\kappa_c}{2G_c}\right)\Omega_{ch}(z) + \beta_b\overline{\theta_{bc}}(z) - \beta_c\overline{\theta_{cc}}(z)}{\frac{\kappa_c}{2G_c} + \frac{1}{2G_b}}$$
$$\Omega_{cc}(z) = \frac{\left[\frac{\kappa_c}{2G_c} - \frac{\kappa_b}{2G_b}\right]\overline{\Phi_{ch}}(z) - \beta_b\overline{\theta_{ba}}(z) + \beta_c\overline{\theta_0}(z)}{\frac{\kappa_b}{2G_b} + \frac{1}{2G_c}} \tag{45b}$$

$$\Phi_0(z) = \frac{\left(\frac{1}{G_b} - \frac{1}{G_a}\right)\left[\overline{\Omega_{ba}}(z-2ih) + 2ih\overline{\Phi'_{ba}}(z-2ih)\right] - 2\beta_b\theta_{bc}(z)}{\frac{1}{G_a} + \frac{\kappa_b}{G_b}}$$
$$\Phi_{aa}(z) = \frac{\frac{\kappa_b+1}{G_b}\Phi_{ba}(z) + 2\beta_b\theta_{ba}(z) - 2\beta_a\theta_{aa}(z)}{\frac{\kappa_a}{G_a} + \frac{1}{G_b}} \tag{46a}$$

$$\Omega_0(z) = 2ih\Phi'_0(z) + \frac{\left(\frac{\kappa_b}{G_b} - \frac{\kappa_a}{G_a}\right)\overline{\Phi_{ba}}(z-2ih) + 2\beta_b\overline{\theta_{ba}}(z-2ih) - 2\beta_a\overline{\theta_{aa}}(z-2ih)}{\frac{\kappa_a}{G_a} + \frac{1}{G_b}}$$
$$\Omega_{aa}(z) = 2ih\Phi'_{aa}(z) + \frac{\frac{\kappa_b+1}{G_b}[\Omega_{ba}(z) - 2ih\Phi'_{ba}(z)] - 2\beta_b\overline{\theta_{bc}}(z-2ih)}{\frac{1}{G_a} + \frac{\kappa_b}{G_b}} \tag{46b}$$

$$\Phi_{ch}(z) = \frac{G_c Q_0 \beta_c}{4\pi k_c(1-v_c)}\log\left(\frac{z-z_s}{z-z_k}\right) \tag{47}$$

$$\Omega_{ch}(z) = \frac{G_c Q_0 \beta_c}{4\pi k_c(1-\nu_c)}\left[\log\left(\frac{z-z_s}{z-z_k}\right) + \frac{z-\overline{z_s}}{z-z_s} - \frac{z-\overline{z_k}}{z-z_k}\right] \tag{48}$$

$$\theta_{ba}(z) = \sum_{n=1}^{\infty}\theta_n(z) \tag{49}$$

$$\theta_{bc}(z) = V_{ab}\sum_{n=1}^{\infty}\overline{\theta_n}(z-2\mathrm{i}h) \tag{50}$$

$$\theta_{cc}(z) = V_{bc}\overline{\theta_0}(z) + U_{cb}V_{ab}\sum_{n=1}^{\infty}\overline{\theta_n}(z-2\mathrm{i}h) \tag{51}$$

$$\theta_{aa}(z) = U_{ab}\sum_{n=1}^{\infty}\theta_n(z) \tag{52}$$

For the special case that material a is made up of material b, Eq. (43) reduces to

$$\Phi(z) = \begin{cases} \dfrac{\frac{1+\kappa_c}{2G_c}\Phi_{ch}(z) + (\beta_c - U_{bc}\beta_b)\theta_0(z)}{\frac{\kappa_b}{2G_b} + \frac{1}{2G_c}} & z \in S_b \\ \dfrac{\left(\frac{1}{2G_c} - \frac{1}{2G_b}\right)\overline{\Omega_{ch}}(z) - \beta_c V_{bc}\overline{\theta_0}(z)}{\frac{\kappa_c}{2G_c} + \frac{1}{2G_b}} + \Phi_{ch} & z \in S_c \end{cases} \tag{53a}$$

$$\Omega(z) = \begin{cases} \dfrac{\frac{1+\kappa_c}{2G_c}\Omega_{ch}(z) - \beta_c V_{bc}\theta_0(z)}{\frac{\kappa_c}{2G_c} + \frac{1}{2G_b}} & z \in S_b \\ \dfrac{\left(\frac{\kappa_c}{2G_c} - \frac{\kappa_b}{2G_b}\right)\overline{\Phi_{ch}}(z) + (\beta_c - U_{bc}\beta_b)\overline{\theta_0}(z)}{\frac{\kappa_b}{2G_b} + \frac{1}{2G_c}} + \Omega_{ch}(z) & z \in S_c \end{cases} \tag{53b}$$

which is in accordance with the exact solution of the bimaterial one (Chao and Shen, 1993).

5. Results and discussion

The thermal potentials as indicated in Eqs. (19) and (22) are expressed in terms of a homogeneous solution $\theta_0(z)$ through the recurrence formulae (20) and (23), respectively. The rate of the convergence depends on the non-dimensional bimaterial constants U_{ab} and V_{ab} (or U_{bc} and V_{bc}). The present series solution converges to the true solution since those bimaterial constants are always less than one. The stress functions as indicated in Eqs. (40) and (43) are expressed in terms of $\Phi_n(z)$ and $\Omega_n(z)$ ($n = 0, 1, 2, \ldots$), which may be calculated from a homogeneous solution $\Phi_0(z)$ and $\Omega_0(z)$ by the recurrence formulae equations (41) and (44). The rate of convergence depends on the ratios $|\Phi_{n+1}(z)|/|\Phi_n(z)|$ and $|\Omega_{n+1}(z)|/|\Omega_n(z)|$, which in turn depend on the non-dimensional bimaterial constants Λ_{ab} and Π_{ab} (or Λ_{cb} and Π_{cb}). For most combinations of materials, Λ and Π are less than 1 and 0.5, respectively, which guarantees rapid convergence. Consequently, the convergence rate becomes more rapid as the differences of the elastic constants of the neighboring materials get smaller. Even though materials a and/or c are rigid or non-existent, the solution remains valid. For another limiting case in which two adjacent materials are identical, the series solution for a trimaterial reduces to the bimaterial one. In order to demonstrate the use of the present approach, the interfacial stresses for a trimaterial and for a film/substrate system are discussed in detail and shown in graphic form. Note that all the calculated results shown in Figs. 2–7 are determined by summing up the first four terms of Eq. (40), since they are checked to achieve a good accuracy with an error less than 0.01% as compared to the first five terms of Eq. (40) for the current problem.

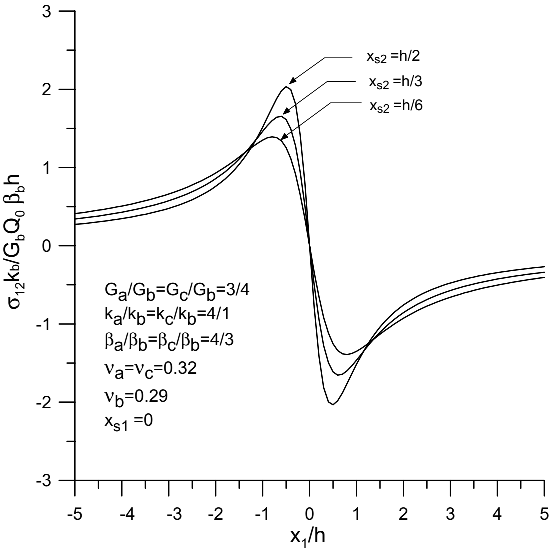

Fig. 2. Interfacial shear stress distribution induced by a point heat source.

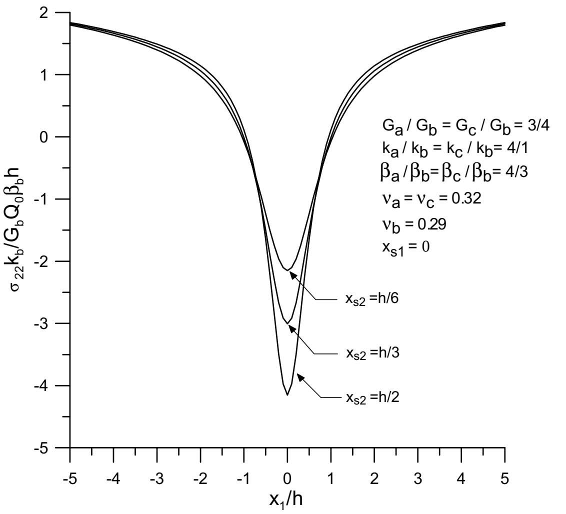

Fig. 3. Interfacial normal stress distribution induced by a point heat source.

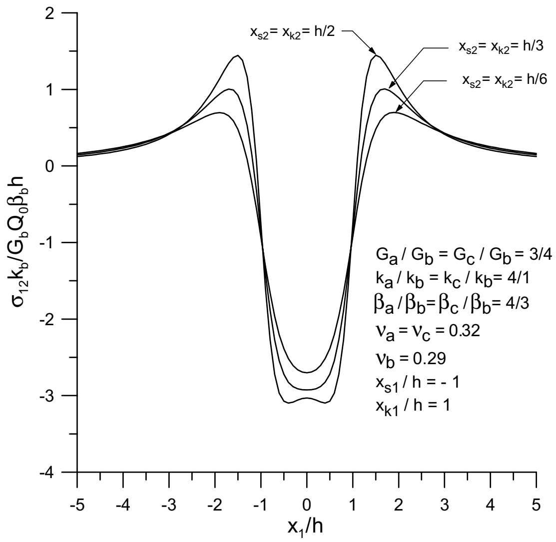

Fig. 4. Interfacial shear stress distribution induced by a pair of point heat source and sink.

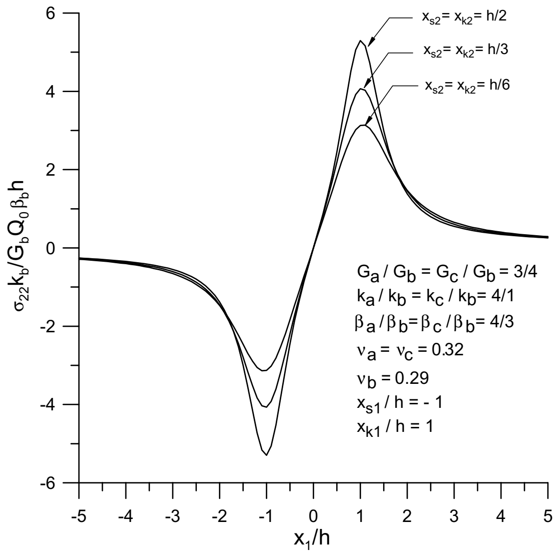

Fig. 5. Interfacial normal stress distribution induced by a pair of point heat source and sink.

5.1. Interfacial stresses for a trimaterial

As our first example, we consider a trimaterial interacted with a point heat source embedded in material *b*. The shear stress and normal stress distributions along the interface between material *a* and material *b* are presented in Figs. 2 and 3, respectively. Both the maximum shear stress and the maximum compressive normal stress decrease with increasing of the distance between the singularity point and the interface. Note

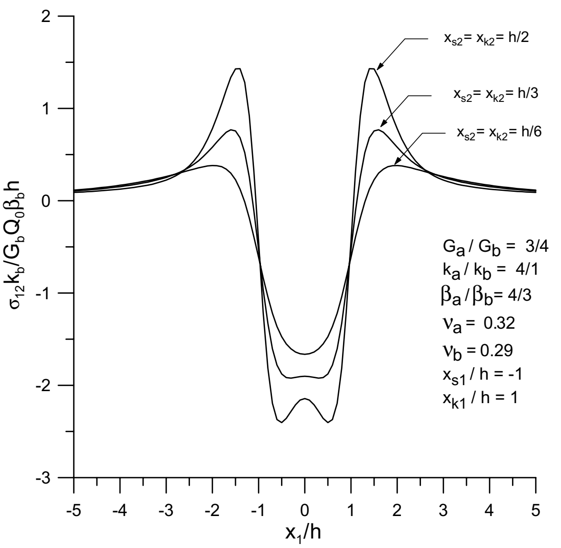

Fig. 6. Interfacial shear stress distribution for a film/substrate structure.

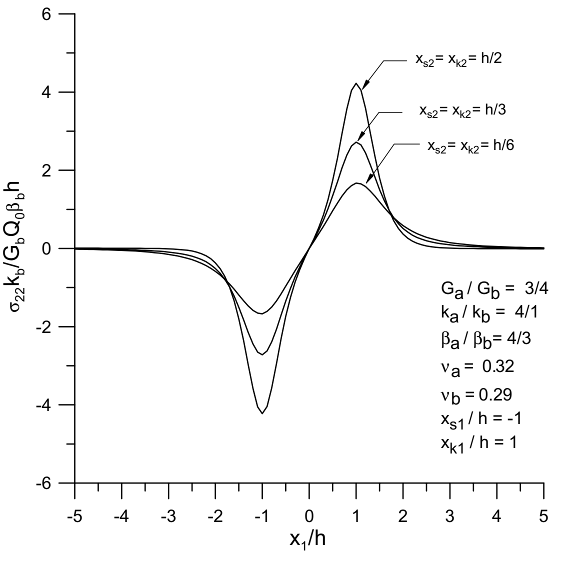

Fig. 7. Interfacial normal stress distribution for a film/substrate structure.

that the normal interfacial stress would not be bounded as $|x_1|$ goes to infinity due to the presence of the singular term $\ln z$ induced by a point heat source. Nevertheless, the interfacial stress remains finite at infinity if one considers the case with a pair of point heat source and sink applied in the material medium as indicated in Figs 4 and 5. Moreover, it is evident that the maximum compressive (or tensile) stress occurs around the location of the heat source (or sink).

5.2. Interfacial stresses for a film/substrate

As a second example we consider a film/substrate structure interacted with a pair of point heat source and sink. When material c (or a) is non-existent, the solution of a film/substrate structure can be obtained by putting $U_{cb} = 2$, $V_{cb} = 1$ in Eq. (19) and $\Lambda_{cb} = \Pi_{cb} = -1$ in Eqs. (40). The distribution of the interfacial stresses between material a and material b is shown in Figs. 6 and 7. It is seen that the trend of the interfacial stresses of the present case is nearly the same as that of the trimaterial one, but the magnitude of interfacial stresses for a film/substrate structure is smaller than that of a trimaterial. This is simply because that the interfacial stresses can be further intensified (or diminished) by the adjacent material having a higher (or lower) stiffness.

6. Conclusion

Thermal stresses for an isotropic trimaterial induced by a point heat source is analyzed in this paper. Within the framework of the procedure of analytical continuation and the method of successive approximations, the solution associated with the heterogeneous problem is sought as transformation on the solution to the corresponding homogeneous problem. Using the present approach, the solution related to the problem consisting of any number of layered medium can also be obtained as the corresponding homogeneous solution is solved. The convergence rate of the series solution depends on the material combinations in such a way that the convergence rate becomes more rapid if the differences of elastic constants of adjacent materials get smaller. The trimaterial solution presented here can be applied to a variety of problems, e.g. a bimaterial, a film/substrate structure, and a finite strip of thin film, etc.

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